

Lecture 15

Fundamentals of Business Mathematics

MATH-114 • BBA 2k26

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Learning Objectives

- Explain an ordinary annuity and its payment timeline.
- Calculate the future value (sum) of an annuity.
- Determine the periodic deposit needed for a future target.
- Calculate the present value of an annuity and the periodic payment.
- Apply present-value calculations to mortgages.
- Evaluate a project using discounted cash flow and net present value.



8.3: What Is an Annuity?

An **annuity** is a series of periodic payments. Examples include regular savings deposits, mortgage payments, and retirement payments.

- In this section, payments are **equal**.
- Each payment is made at the **end** of a compounding period.
- This is an **ordinary annuity**; a payment made at the end of a period starts earning interest in the next period.



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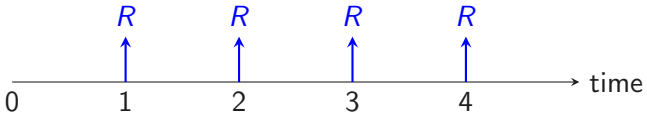



Sum (Future Value) of an Annuity

Let R be the payment at the end of each period, i the interest rate per period, and n the number of payments.

$$S_n = R + R(1+i) + R(1+i)^2 + \cdots + R(1+i)^{n-1}.$$



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Using the geometric-series formula:

$$\boxed{S_n = R\left[\frac{(1+i)^n - 1}{i}\right]}$$

- The bracketed quantity is the **series compound amount factor**.
- S_n is the amount immediately after the nth payment.
- When annual rate is j and payments are m times per year: $i = j/m,\ n = mt$.



Example 1

A person plans to deposit \$1,000 in a tax-exempt savings plan at the end of this year and an equal sum at the end of each following year. If interest is expected to be earned at the rate of 6 percent per year compounded annually, to what sum will the investment grow at the time of the fourth deposit?



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Example 1: Solution by Accumulating Deposits

At the time of the fourth deposit:

$$\begin{aligned} S_4 &= 1000(1.06)^3 + 1000(1.06)^2 \\ &\quad + 1000(1.06) + 1000 \\ &= 1191.016 + 1123.60 + 1060 + 1000 \\ &= \boxed{\$4,374.62}. \end{aligned}$$



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Interpretation: Total deposits are \$4,000; the remaining \$374.62 is interest.



Example 2

Re-solve the Example 1 using formula.



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Example 2: Solution Using the Annuity Formula

$$R = \$1,000, \quad i = 0.06, \quad n = 4.$$

$$S_4 = 1000\left[\frac{(1.06)^4 - 1}{0.06}\right] = 1000(4.374616)$$

$$\boxed{S_4 \approx \$4,374.62}.$$



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Lesson

The annuity formula replaces a long sum with one calculation. It produces the same result as accumulating the individual deposits.



Example 3

A teenager plans to deposit \$50 in a savings account at the end of each quarter for the next 6 years. Interest is earned at a rate of 8 percent per year compounded quarterly. What should her account balance be 6 years from now? How much interest will she earn?



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Determine the quarterly rate, number of quarterly deposits, and total amount deposited.



Example 3: Solution

$$R = 50, \qquad i = \frac{0.08}{4} = 0.02, \qquad n = 6(4) = 24.$$

$$S_{24} = 50\left[\frac{(1.02)^{24} - 1}{0.02}\right] \approx \boxed{\$1,521.09}.$$



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$$R = 50, \qquad i = \frac{0.08}{4} = 0.02, \qquad n = 6(4) = 24.$$

$$S_{24} = 50\left[\frac{(1.02)^{24} - 1}{0.02}\right] \approx \boxed{\$1,521.09}.$$

Total deposits:

$$24(50) = \$1,200.$$

Interest earned:

$$\$1,521.09 - \$1,200 = \boxed{\$321.09}.$$



Determining the Size of an Annuity

Suppose the future target S_n is known, but the required periodic deposit R is unknown.

$$S_n = R\left[\frac{(1+i)^n - 1}{i}\right]$$



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$$S_n = R\left[\frac{(1+i)^n - 1}{i}\right]$$

Solving for R:

$$\boxed{R = S_n\left[\frac{i}{(1+i)^n - 1}\right]}.$$

- The bracketed expression is the **sinking fund factor**.
- A **sinking fund** is a series of deposits intended to accumulate a future sum.



Example 4

A corporation wants to establish a sinking fund beginning at the end of this year. Annual deposits will be made at the end of this year and for the following 9 years. If deposits earn interest at the rate of 8 percent per year compounded annually, how much money must be deposited each year in order to have \$12 million at the time of the 10th deposit? How much interest will be earned?



Example 4

A corporation wants to establish a sinking fund beginning at the end of this year. Annual deposits will be made at the end of this year and for the following 9 years. If deposits earn interest at the rate of 8 percent per year compounded annually, how much money must be deposited each year in order to have \$12 million at the time of the 10th deposit? How much interest will be earned? **Identify:** S_{10}, i, n, and the unknown payment R.



Example 4: Solution

$$S_{10} = 12{,}000{,}000, \qquad i = 0.08, \qquad n = 10.$$

$$R = 12{,}000{,}000 \left[\frac{0.08}{(1.08)^{10} - 1} \right] \approx \$828{,}353.86.$$



Example 4: Solution

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$$R = 12{,}000{,}000 \left[\frac{0.08}{(1.08)^{10} - 1} \right] \approx \$828{,}353.86.$$

Textbook table approach: Using the rounded sinking fund factor 0.06903, the book obtains $R = \$828{,}360$. With that rounded payment:

$$\text{Deposits} = 10(828{,}360) = \$8{,}283{,}600,$$

$$\text{Interest} \approx 12{,}000{,}000 - 8{,}283{,}600 = \boxed{\$3{,}716{,}400}.$$



Example 5

Question

Assume in the last example that the corporation is going to make quarterly deposits and that interest is earned at the rate of 8 percent per year compounded quarterly. How much money should be deposited each quarter? How much less will the company have to deposit over the 10-year period as compared with annual deposits and annual compounding?



Example 5

Question

Assume in the last example that the corporation is going to make quarterly deposits and that interest is earned at the rate of 8 percent per year compounded quarterly. How much money should be deposited each quarter? How much less will the company have to deposit over the 10-year period as compared with annual deposits and annual compounding?

- How many deposits will be made?
- What interest rate applies to each quarter?
- What total deposits should be compared?



8.4: Present Value of an Ordinary Annuity

Sometimes we know the periodic payments and want their **equivalent value today**. If R is paid at the end of each period:

$$A_n = \frac{R}{1+i} + \frac{R}{(1+i)^2} + \cdots + \frac{R}{(1+i)^n}.$$



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$$A_n = \frac{R}{1+i} + \frac{R}{(1+i)^2} + \cdots + \frac{R}{(1+i)^n}.$$

Using the geometric-series formula:

$$\boxed{A_n = R\left[\frac{1-(1+i)^{-n}}{i}\right]}.$$

The bracketed expression is the **series present value factor**.



Example 6: Present Value of Payments

Question

A company will receive \$6,000 at the end of each year for 5 years. If the appropriate interest rate is 7% per year, what is the present value of these receipts?



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Solution

$$R = 6000, \quad i = 0.07, \quad n = 5,$$

$$A_5 = 6000 \left[\frac{1 - (1.07)^{-5}}{0.07} \right] \approx \boxed{\$24,601.18}.$$



Example 6: Present Value of Payments

Question

A company will receive \$6,000 at the end of each year for 5 years. If the appropriate interest rate is 7% per year, what is the present value of these receipts?

Solution

$$R = 6000, \quad i = 0.07, \quad n = 5,$$

$$A_5 = 6000 \left[\frac{1 - (1.07)^{-5}}{0.07} \right] \approx \boxed{\$24,601.18}.$$

The five future payments are equivalent to approximately \$24,601.18 today at a 7% discount rate.



Determining the Size of an Annuity: Present Value

If a loan or present amount A_n is known, solve for the periodic payment:

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$$\boxed{R = A_n\left[\frac{i}{1-(1+i)^{-n}}\right]}.$$

- The bracketed expression is the **capital-recovery factor**.
- This formula is used for equal payments that repay a loan over a fixed number of periods.



Example 7: Loan Repayment

Question

A business borrows \$50,000 and agrees to repay it in equal annual payments over 5 years. If the interest rate is 8% per year, what annual payment is required?



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$$A_5 = 50{,}000, \quad i = 0.08, \quad n = 5.$$

$$R = 50{,}000 \left[\frac{0.08}{1 - (1.08)^{-5}} \right] \approx \boxed{\$12,522.82}.$$



Mortgages: Monthly Payment Model

A mortgage is a loan commonly repaid by a series of regular payments. If the loan principal is A_n, the nominal annual rate is j, and there are m payments per year for t years:

$$i = \frac{j}{m}, \qquad n = mt,$$

$$\boxed{R = A_n \left[\frac{i}{1 - (1+i)^{-n}} \right]}.$$



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Important

Use the *amount borrowed*, not the total property price, as the present value of mortgage payments. This model assumes a fixed rate and equal end-of-period payments.



Example 8: Mortgage Payment

Question

A buyer borrows \$180,000 through a 20-year mortgage at a nominal annual rate of 6%, with interest compounded monthly. Determine the monthly payment (excluding taxes and insurance).



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$$A_n = 180{,}000, \qquad i = 0.06/12 = 0.005, \qquad n = 20(12) = 240.$$



Example 8: Solution

$$R = 180{,}000\left[\frac{0.005}{1-(1.005)^{-240}}\right] \approx \boxed{\$1,289.58 \text{ per month}}.$$



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Interpretation

Each monthly payment covers interest on the outstanding loan and repayment of part of the principal.



8.5: Cost–Benefit Analysis

A business project often involves:

- an initial cash outflow (investment);
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Because cash flows occur at different times, compare them on a **present-value basis**.

$$\boxed{\text{NPV} = \text{PV of cash inflows} - \text{PV of cash outflows}}$$

- If NPV > 0, the discounted benefits exceed discounted costs.
- If NPV < 0, discounted costs exceed discounted benefits.
- NPV $= 0$ indicates equality at the selected discount rate.



Discounted Cash Flow: The Basic Idea

For an individual cash flow C_t received t periods from now:

$$PV(C_t) = \frac{C_t}{(1+i)^t}.$$

For a project with initial cost C_0 and future net cash flows $C_1, \ldots, C_n$:

$$\boxed{\text{NPV} = -C_0 + \sum_{t=1}^{n} \frac{C_t}{(1+i)^t}}.$$



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Method: List cash flows by date, discount each to today, and add the present values with their appropriate signs.



Example 9: Cost–Benefit Decision

Question

A company is considering equipment costing \$30,000 today. It expects net cash benefits of \$12,000 at the end of each of the next three years. If the discount rate is 8% annually, calculate the project's net present value and state whether the discounted benefits exceed the cost.



Example 9: Cost–Benefit Decision

Question

A company is considering equipment costing \$30,000 today. It expects net cash benefits of \$12,000 at the end of each of the next three years. If the discount rate is 8% annually, calculate the project's net present value and state whether the discounted benefits exceed the cost. **Think:** Is the three-year benefit stream an ordinary annuity?



Example 9: Solution

Present value of benefits:

$$PV = 12{,}000\left[\frac{1-(1.08)^{-3}}{0.08}\right] \approx \$30{,}925.16.$$



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Net present value:

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Interpretation

At an 8% discount rate, the discounted benefits exceed the initial cost by about \$925.16. On this financial criterion alone, the project has positive NPV.



Which Formula Should You Use?

Question asks for ...	**Formula**
Future sum of equal deposits	$S_n = R[(1+i)^n - 1]/i$
Deposit to reach a future target	$R = S_n i/[(1+i)^n - 1]$
Present value of equal receipts	$A_n = R[1-(1+i)^{-n}]/i$
Loan or mortgage payment	$R = A_n i/[1-(1+i)^{-n}]$
Value of an investment project	NPV = PV benefits − PV costs



Review and Discussion

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1. Why does the first payment of an ordinary annuity earn more interest than the final payment?
2. Why is the present value of future payments less than their undiscounted total when $i > 0$?
3. What changes in a sinking-fund calculation when deposits become quarterly rather than annual?
4. Why should future project benefits be discounted before comparing them with a cost today?

