

Lecture 16

Fundamentals of Business Mathematics

MATH-114 • BBA 2k26

Dr. Rizwan Jahangir
NUST Business School
National University of Sciences & Technology



Learning Objectives

- Explain an ordinary annuity and its payment timeline.
- Calculate the future value (sum) of an annuity.
- Determine the periodic deposit needed for a future target.
- Calculate the present value of an annuity and the periodic payment.
- Apply present-value calculations to mortgages.
- Evaluate a project using discounted cash flow and net present value.



8.4: Present Value of an Ordinary Annuity

Sometimes we know the periodic payments and want their **equivalent value today**. If R is paid at the end of each period:

$$A_n = \frac{R}{1+i} + \frac{R}{(1+i)^2} + \cdots + \frac{R}{(1+i)^n}.$$

Using the geometric-series formula:

$$\boxed{A_n = R\left[\frac{1-(1+i)^{-n}}{i}\right]}.$$

The bracketed expression is the **series present value factor**.



Example 6: Present Value of Payments

Question

A company will receive \$6,000 at the end of each year for 5 years. If the appropriate interest rate is 7% per year, what is the present value of these receipts?

Solution

$$R = 6000, \quad i = 0.07, \quad n = 5,$$

$$A_5 = 6000 \left[\frac{1 - (1.07)^{-5}}{0.07} \right] \approx \boxed{\$24,601.18}.$$

The five future payments are equivalent to approximately \$24,601.18 today at a 7% discount rate.



Determining the Size of an Annuity: Present Value

If a loan or present amount A_n is known, solve for the periodic payment:

$$A_n = R\left[\frac{1-(1+i)^{-n}}{i}\right].$$

$$\boxed{R = A_n\left[\frac{i}{1-(1+i)^{-n}}\right]}.$$

- The bracketed expression is the **capital-recovery factor**.
- This formula is used for equal payments that repay a loan over a fixed number of periods.



Example 7: Loan Repayment

Question

A business borrows \$50,000 and agrees to repay it in equal annual payments over 5 years. If the interest rate is 8% per year, what annual payment is required?

Solution

$$A_5 = 50{,}000, \quad i = 0.08, \quad n = 5.$$

$$R = 50{,}000 \left[\frac{0.08}{1 - (1.08)^{-5}} \right] \approx \boxed{\$12,522.82}.$$



Mortgages: Monthly Payment Model

A mortgage is a loan commonly repaid by a series of regular payments. If the loan principal is A_n, the nominal annual rate is j, and there are m payments per year for t years:

$$i = \frac{j}{m}, \qquad n = mt,$$

$$\boxed{R = A_n \left[\frac{i}{1 - (1+i)^{-n}} \right]}.$$

Important

Use the *amount borrowed*, not the total property price, as the present value of mortgage payments. This model assumes a fixed rate and equal end-of-period payments.



Example 8: Mortgage Payment

Question

A buyer borrows \$180,000 through a 20-year mortgage at a nominal annual rate of 6%, with interest compounded monthly. Determine the monthly payment (excluding taxes and insurance).

Set up

$$A_n = 180{,}000, \qquad i = 0.06/12 = 0.005, \qquad n = 20(12) = 240.$$



Example 8: Solution

$$R = 180{,}000\left[\frac{0.005}{1-(1.005)^{-240}}\right] \approx \boxed{\$1,289.58 \text{ per month}}.$$

Interpretation

Each monthly payment covers interest on the outstanding loan and repayment of part of the principal.



8.5: Cost–Benefit Analysis

A business project often involves:

- an initial cash outflow (investment);
- future cash inflows (benefits);
- possibly future cash outflows (costs).

Because cash flows occur at different times, compare them on a **present-value basis**.

$$\boxed{\text{NPV} = \text{PV of cash inflows} - \text{PV of cash outflows}}$$

- If NPV > 0, the discounted benefits exceed discounted costs.
- If NPV < 0, discounted costs exceed discounted benefits.
- NPV $= 0$ indicates equality at the selected discount rate.



Discounted Cash Flow: The Basic Idea

For an individual cash flow C_t received t periods from now:

$$PV(C_t) = \frac{C_t}{(1+i)^t}.$$

For a project with initial cost C_0 and future net cash flows $C_1, \ldots, C_n$:

$$\boxed{\text{NPV} = -C_0 + \sum_{t=1}^{n} \frac{C_t}{(1+i)^t}}.$$

Method: List cash flows by date, discount each to today, and add the present values with their appropriate signs.



Example 9: Cost–Benefit Decision

Question

A company is considering equipment costing \$30,000 today. It expects net cash benefits of \$12,000 at the end of each of the next three years. If the discount rate is 8% annually, calculate the project's net present value and state whether the discounted benefits exceed the cost.

Think: Is the three-year benefit stream an ordinary annuity?



Example 9: Solution

Present value of benefits:

$$PV = 12{,}000\left[\frac{1-(1.08)^{-3}}{0.08}\right] \approx \$30{,}925.16.$$

Net present value:

$$\text{NPV} = 30{,}925.16 - 30{,}000 = \boxed{\$925.16}.$$

Interpretation

At an 8% discount rate, the discounted benefits exceed the initial cost by about \$925.16. On this financial criterion alone, the project has positive NPV.



Which Formula Should You Use?

Question asks for ...	**Formula**
Future sum of equal deposits	$S_n = R[(1+i)^n - 1]/i$
Deposit to reach a future target	$R = S_n i/[(1+i)^n - 1]$
Present value of equal receipts	$A_n = R[1-(1+i)^{-n}]/i$
Loan or mortgage payment	$R = A_n i/[1-(1+i)^{-n}]$
Value of an investment project	NPV = PV benefits − PV costs



Review and Discussion

1. Why does the first payment of an ordinary annuity earn more interest than the final payment?
2. Why is the present value of future payments less than their undiscounted total when $i > 0$?
3. What changes in a sinking-fund calculation when deposits become quarterly rather than annual?
4. Why should future project benefits be discounted before comparing them with a cost today?



Suggested Problems: Section 8.5

Section 8.5: Cost–Benefit Analysis

- Problems 1–2: NPV with equal annual cash flows (Figures 8.14–8.15).
- Problems 3–6: NPV of investment projects (Figures 8.16–8.19).

For each problem:

- Compute the net present value (NPV).
- Determine whether the investment satisfies the minimum desired rate of return criterion.

Additional Challenge (Optional)

- Problems 7–8: Estimating the actual rate of return using the trial-and-error approach.



Additional Practice Problems: Cost–Benefit Analysis

1. A company is considering purchasing a machine for \$60,000. The machine is expected to generate net annual cash benefits of \$18,000 at the end of each year for 5 years. If the annual discount rate is 10%, determine the net present value and decide whether the investment is financially worthwhile.

2. A business invests \$40,000 in a project that is expected to generate net cash inflows of \$12,000, \$15,000, and \$20,000 at the end of the next three years, respectively. Determine its net present value using an annual discount rate of 8%.

3. A company is considering two mutually exclusive projects, each requiring an initial investment of \$50,000. Project A generates \$20,000 annually for 3 years, whereas Project B generates \$16,000 annually for 4 years. All receipts occur at the end of each year. Using a discount rate of 8%, determine which project has the higher net present value.



Additional Practice: Solutions (1/2)

Problem 1: Equal Annual Cash Flows

$$PV = 18,000 \left[\frac{1 - (1.10)^{-5}}{0.10} \right] = \$68,234.16.$$

$$\text{NPV} = 68,234.16 - 60,000 = \boxed{\$8,234.16}.$$

Since $\text{NPV} > 0$, the investment is financially worthwhile at the given discount rate.

Problem 2: Unequal Annual Cash Flows

Since the cash inflows are unequal, discount each cash flow separately.

$$PV = \frac{12,000}{1.08} + \frac{15,000}{(1.08)^2} + \frac{20,000}{(1.08)^3}$$

$$PV \approx \$39,847.58.$$

$$\text{NPV} = 39,847.58 - 40,000 = \boxed{-\$152.42}.$$

The discounted benefits are less than the initial cost.



Additional Practice: Solutions (2/2)

Problem 3: Comparison of Investment Projects

Project A:

$$PV_A = 20,000 \left[\frac{1 - (1.08)^{-3}}{0.08} \right] = \$51,541.94.$$

$$\text{NPV}_A = 51,541.94 - 50,000 = \boxed{\$1,541.94}.$$

Project B:

$$PV_B = 16,000 \left[\frac{1 - (1.08)^{-4}}{0.08} \right] = \$52,994.03.$$

$$\text{NPV}_B = 52,994.03 - 50,000 = \boxed{\$2,994.03}.$$

Decision

Both projects have positive net present values. However, Project B is preferred under the NPV