

Lecture 18

Fundamentals of Business Mathematics

MATH-114 • BBA 2k26

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Lecture 18 Outline: Matrix Multiplication

- Inner product of vectors
- Compatibility and dimensions of matrix products
- Row-by-column computation
- Properties of multiplication
- Representation of equations and systems



Recap: Dimensions and Transpose

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} -4 \\ 2 \end{pmatrix}.$$



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$$A = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} -4 \\ 2 \end{pmatrix}.$$

A is 2×2 and B is 2×1. Addition is not defined. Today we will see why multiplication AB is defined.



The Inner Product

For a row vector $u = (u_1, \ldots, u_n)$ and a column vector $v = (v_1, \ldots, v_n)^T$,

$$u \cdot v = u_1 v_1 + \cdots + u_n v_n.$$



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Multiply corresponding elements and add the products. The vectors must have the same number of elements.

$$\begin{pmatrix} 5 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \end{pmatrix} = 5(4) - 2(6) = 8.$$



Example 1: Inner Product

Given the row and column vectors

$$M = \begin{pmatrix} 5 & -2 & 0 & 1 & 3 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} -2 \\ -4 \\ 10 \\ 20 \\ 6 \end{pmatrix}$$

Find the inner product.



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Find the inner product.

$$M \cdot N = 5(-2) + (-2)(-4) + 0(10) + 1(20) + 3(6) = \boxed{36}.$$



Matrix Multiplication: Compatibility

For A of dimension $m \times n$ and B of dimension $p \times q$, AB exists if and only if $n = p$.



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- $(2 \times 3)(3 \times 4)$ gives 2×4.
- $(3 \times 2)(3 \times 4)$ is undefined.



Row-by-Column Rule

If $C = AB$, then c_{ij} is the inner product of row i of A and column j of B.

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$



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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$



Example 2: Matrix Product

To find the matrix product AB, where

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The dimensions $(2 \times 2)(2 \times 1)$ give 2×1.

$$AB = \begin{pmatrix} 2(-4) + 4(2) \\ 3(-4) + 1(2) \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ -10 \end{pmatrix}}.$$



Example 3: Two Square Matrices

Classroom example: Find AB for

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix}.$$



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$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 2(4)+1(1) & 2(0)+1(2) \\ 3(4)+4(1) & 3(0)+4(2) \end{pmatrix} = \boxed{\begin{pmatrix} 9 & 2 \\ 16 & 8 \end{pmatrix}}.$$



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$$BA = \begin{pmatrix} 4(2)+0(3) & 4(1)+0(4) \\ 1(2)+2(3) & 1(1)+2(4) \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 8 & 9 \end{pmatrix}.$$



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Thus $AB \neq BA$. Matrix multiplication is not generally commutative. In some cases only one order is even defined.



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Whenever the indicated operations are defined:

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Whenever the indicated operations are defined:

- Associative: $A(BC) = (AB)C$.
- Distributive: $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$.
- Identity: $AI = A$ and $IA = A$ for compatible identity matrices.



Representation of an Equation

The expression $3x_1 - 5x_2$ is an inner product:

$$\begin{pmatrix} 3 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$



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Consequently the equation $3x_1 - 5x_2 = 20$ can be written as

$$\begin{pmatrix} 3 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 20 \end{pmatrix}.$$



Representation of Systems

A system of m linear equations in n variables can be represented as $AX = B$:

$$\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}_{B}.$$



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A: coefficient matrix ($m \times n$); X: variable vector ($n \times 1$); B: constants vector ($m \times 1$).



Example 4: Matrix Representation

The system of equations

$$\begin{aligned} x_1 - 2x_2 + 3x_4 + x_5 &= 100 \\ x_1 - 3x_3 + x_4 &= 60 \\ 4x_2 - x_3 + 2x_4 + x_5 &= 125 \end{aligned}$$

can be represented in the matrix form $AX = B$.



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can be represented in the matrix form $AX = B$.

$$\begin{pmatrix} 1 & -2 & 0 & 3 & 1 \\ 1 & 0 & -3 & 1 & 0 \\ 0 & 4 & -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 100 \\ 60 \\ 125 \end{pmatrix}.$$



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Include zero coefficients for missing variables.



Example 5: Nonlinear Equation

Matrix representation of equations is not restricted to linear equations. The quadratic equation

$$10x^2 - 4x + 50 = 0$$

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Matrix representation of equations is not restricted to linear equations. The quadratic equation

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can be represented by the equivalent matrix equation

$$\begin{pmatrix} 10 & -4 & 50 \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}.$$

This representation does not make the equation linear in x.



Additional Practice Problems

A company sells three products at two stores. The quantity matrix and price vector are

$$Q = \begin{pmatrix} 20 & 15 & 10 \\ 12 & 18 & 25 \end{pmatrix}, \qquad P = \begin{pmatrix} 5 \\ 8 \\ 12 \end{pmatrix}.$$

1. Determine the dimension of QP and calculate it. Interpret the result.
2. Is PQ defined? Explain.
3. Write the production constraints $2x + 3y = 120$, $4x + y = 100$ in matrix form.



Additional Practice: Solutions

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$$QP = \begin{pmatrix} 20(5) + 15(8) + 10(12) \\ 12(5) + 18(8) + 25(12) \end{pmatrix} = \begin{pmatrix} 340 \\ 504 \end{pmatrix}.$$

These are the two stores' revenues in the given currency units.



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$$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 120 \\ 100 \end{pmatrix}.$$



Suggested Textbook Problems

Section 9.3: Matrix Operations (remaining part)

- Problems 7–16: Inner products and compatibility.
- Problems 17–36: Matrix multiplication wherever possible.
- Problems 37–46: Representation of systems in matrix form.
- Problem 47: Associative and distributive properties.

